Head-on collision of dust-acoustic solitary waves

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The head-on collision between two dust-acoustic solitary waves in an unmagnetized dusty plasma is investigated, including consideration of the dust charge variation. By using the extended Poincaré-Lighthill-Kuo perturbation method, the analytical phase shifts following the head-on collision are derived. The effects of the ratio of ion temperature to electron temperature, the ratio of the number density of ions to the number density of electrons, and the dust charge variation on the phase shift are studied. It is found that the dust charge variation significantly modifies the phase shift.

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It is well known that the dynamics of dusty plasmas is very different from that of the usual electron-ion plasma. The presence of highly negatively charged dust grains in plasma can modify the collective behavior of a plasma, as well as excite new modes and new nonlinear phenomena, such as dust-ion-acoustic (DIA) waves [1], dust-acoustic (DA) waves [2], dust lattice (DL) waves [3], dust Coulomb waves [4], dust voids [5,6], dust vortex flows [7], etc. Among them, the DIA solitary waves, the DA solitary waves, the DL solitary waves, and the envelope DIA/DA solitary waves [8] are very important nonlinear structures and have been extensively studied both theoretically and experimentally and reported in some review articles [9–11] or books [12,13] over the last few years. In recent years, investigation of the DIA/DA solitary waves has also been extended to nonplanar geometry [14–18] and resulted in considerable success in clarifying many aspects of the characteristics of DIA/DA solitary waves. In the process of soliton propagation in dusty plasmas, wave-wave interaction is another interesting and important nonlinear phenomenon. Up to now, however, there is no detailed investigation about the interaction of DIA/DA solitary waves. We know that one of the striking properties of solitons is their asymptotic preservation of form when they undergo a collision, as first remarked by Zabusky and Kruskal [19]. In a one- (or quasi-one-)dimensional system, there are two distinct soliton interactions. One is the overtaking collision and the other is the head-on collision [20]. Because of the multisoliton solutions (travel in the same direction) of the Korteweg–de Vries (KdV) equation, the overtaking collision of solitary waves can be studied by the inverse scattering transformation method [21]. However, for a head-on collision between two solitary waves, we must search for the evolution of waves traveling to both sides, and hence we need to employ a suitable asymptotic expansion to solve the original fluid dynamic equations. Hence, in this paper, based on a one-dimensional model, the head-on collision between two DA solitary waves is investigated by the extended Poincaré-Lighthill-Kuo (PLK) method [20,22,23]. It is shown that bi-directional solitary waves are propagated and head-on collision of these two solitons occurs. The phase shifts and the trajectories of the two solitons after collision are obtained. The effect of dust charge variation, which significantly affects the characteristic collective motion of plasma, on the properties of head-on collisions of DA solitary waves is also included. We expect that the present investigation will provide insight into and understanding of the collective phenomenon related to DA solitary wave collisions that are of vital importance in laboratory plasmas as well as in plasma applications.

Here, we consider an unmagnetized collisionless plasma consisting of ions, electrons, and cold, extremely massive, microsized, negatively charged dust fluid. The basic equations governing the dusty plasmas for this system are [24]

$$\frac{\partial n_d}{\partial t}+\frac{\partial (n_d u_d)}{\partial x}=0, \tag{1}$$

$$\frac{\partial u_d}{\partial t}+u_d\frac{\partial u_d}{\partial x}=Z_d\frac{\partial \phi}{\partial x}, \tag{2}$$

$$\frac{\partial^2 \phi}{\partial x^2}=Z_d n_d+n_e-n_i; \tag{3}$$

the ion number density n_i and electron number density n_e satisfy the Boltzmann distribution,

$$n_i=\mu \exp(-s\phi), \quad n_e=\nu \exp(\beta s\phi). \tag{4}$$

In the above equations, $\beta=T_i/T_e$ is the ratio of ion temperature to electron temperature and $\delta=n_{i0}/n_{e0}$ is the ratio of the number density of ions to the number density of electrons, $s=1/(\mu+\nu\beta)$, $\mu=\delta/(\delta-1)$, $\nu=1/(\delta-1)$, and the other notation has its usual meaning. The densities of electrons and ions are normalized by $Z_{d0}n_{d0}$. The variables t, x, n_d, u_d, and ϕ are normalized by $\omega_d^{-1}=(m_d/4\pi n_{d0}Z_{d0}^2e^2)^{1/2}$, $\lambda_{Dd}=(T_{eff}/4\pi Z_{d0}n_{d0}e^2)^{1/2}$, n_{d0}, $C_d=(Z_{d0}T_{eff}/m_d)^{1/2}$, and T_{eff}/e, respectively, where n_{i0}, n_{e0}, and n_{d0} are the unperturbed number densities of ions, electrons, and dust particles, respectively, Z_{d0} is the unperturbed number of charges residing on the dust particles measured in units of the electron charge, and $T_{eff}=T_iT_e/(\mu T_e+\nu T_i)$.

The dust charge variation Q_d comes in through the charge current balance equation [24,25]

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$$\frac{\partial Q_d}{\partial t}+u_d\frac{\partial Q_d}{\partial x}=I_e+I_i, \tag{5}$$

which is valid for grain charging arising from plasma currents due to electrons and ions reaching the grain surface. When the streaming velocities of the electrons and ions are much smaller than their corresponding thermal velocities, the expressions for the electron and ion currents for spherical grains of radius R are given by $I_e=-\pi R^2 e(8T_e/\pi m_e)^{1/2}n_e\exp(e\Phi/T_e),I_i=\pi R^2 e(8T_i/\pi m_i)^{1/2}n_i(1-e\Phi/T_i)$, where Φ denotes the dust grain surface potential relative to the plasma potential. It is well known that the characteristic time for dust motion is of the order of tens of milliseconds for micrometer-sized grains, while the dust charging time is typically of the order of 10^{-8} s; therefore, the motion of dust is not so fast that the contribution from the electron current to the dust is balanced by the ions. It follows that $dQ_d/dt \ll I_e, I_i$, the current balance equation reads $I_e+I_i=0$, and we obtain

$$\alpha\delta(1-s\psi)\exp(-s\phi)-\exp[\beta s(\phi+\psi)]=0, \tag{6}$$

where $\alpha=\sqrt{\beta/\mu_i}$, $\mu_i=m_i/m_e$, and $\psi=e\Phi/T_{eff}$. Because the dust charge $Q_d=R\Phi$, i.e., $-eZ_d=RT_{eff}\psi/e$, then we obtain the normalized dust charge (also denoted by Z_d)

$$Z_d=\frac{\psi}{\psi_0}, \tag{7}$$

where $\psi_0=\psi(\phi=0)$ is the dust surface floating potential corresponding to the unperturbed plasma potential, which is determined by

$$\alpha\delta(1-s\psi_0)-\exp(\beta s\psi_0)=0. \tag{8}$$

Now, according to Eq. (6) and after expanding $\psi=\psi(0+\phi)$ near $\phi=0$, Eq. (7) becomes

$$Z_d=1+\gamma_1\phi+\gamma_2\phi^2+\cdots, \tag{9}$$

where

$$\gamma_1=\frac{1}{\psi_0}\frac{d\psi(\phi)}{d\phi}\bigg|_{\phi=0}=-\frac{1}{\psi_0}\frac{(1+\beta)(1-s\psi_0)}{1+\beta(1-s\psi_0)}, \tag{10}$$

$$\gamma_2=\frac{1}{2\psi_0}\frac{d^2\psi(\phi)}{d\phi^2}\bigg|_{\phi=0}=\frac{\psi_0^2}{2}\frac{s\gamma_1^3}{(1+\beta)(1-s\psi_0)^2}. \tag{11}$$

The variables γ_1,γ_2, which are related to the physical parameters, describe the effects of dust charge variation. Setting $\gamma_1=\gamma_2=0$, then we obtain the constant dust charge case, i.e., $Z_d=1$.

We assume two solitons A and B in the plasma, which are, asymptotically, far apart in the initial state and travel toward each other. After some time they interact, collide, and then depart. We also assume that solitons have small amplitudes $\sim\epsilon$ (where ϵ is a smallness formal perturbation parameter) and the interactions between solitons are weak. Hence, we expect that the collision will be quasielastic, so it will cause only shifts of the postcollision trajectories (phase shift). In order to analyze the effects of collision, we employ an extended PLK perturbation method [20,22,23]. This extended PLK method is a combination of the standard reductive perturbation method [26,27] with the technique of strained coordinates. The main idea of this perturbation method is as follows. In the limit of the long wavelength approximation, asymptotic expansions for both the flow field variables and the spatial or time coordinates are used. This makes a uniformly valid asymptotic expansion (i.e., eliminates secular terms) and at the same time obtains the change of the trajectories (i.e., phase shifts) of the solitary waves after the collision. According to this method, we introduce the stretched coordinates

$$\xi=\epsilon(x-\lambda t)+\epsilon^2P_0(\eta,\tau)+\epsilon^3P_1(\xi,\eta,\tau)+\cdots, \tag{12}$$

$$\eta=\epsilon(x+\lambda t)+\epsilon^2Q_0(\xi,\tau)+\epsilon^3Q_1(\xi,\eta,\tau)+\cdots, \tag{13}$$

$$\tau=\epsilon^3 t, \tag{14}$$

where ξ and η denote the trajectories of the two solitons traveling to the right and left, respectively. The wave velocity λ and the variables P_j and Q_j are to be determined. From Eqs. (12)–(14), we have

$$\frac{\partial}{\partial x}=\epsilon\left(\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}\right)+\epsilon^3\left(P_{0\eta}\frac{\partial}{\partial\xi}+Q_{0\xi}\frac{\partial}{\partial\eta}\right)+\cdots, \tag{15}$$

$$\frac{\partial}{\partial t}=\epsilon\lambda\left(-\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}\right)+\epsilon^3\left(\frac{\partial}{\partial\tau}+\lambda P_{0\eta}\frac{\partial}{\partial\xi}-\lambda Q_{0\xi}\frac{\partial}{\partial\eta}\right)+\cdots. \tag{16}$$

Introducing the asymptotic expansion

$$n_d=1+\epsilon^2n_1+\epsilon^3n_2+\cdots, \tag{17}$$

$$u_d=\epsilon^2u_1+\epsilon^3u_2+\cdots, \tag{18}$$

$$\phi=\epsilon^2\phi_1+\epsilon^3\phi_2+\cdots. \tag{19}$$

Then, substituting Eqs. (4) and (9) and Eqs. (15)–(19) into Eqs. (1)–(3) and by equating the powers of ϵ, we obtain a hierarchy of coupled equations in different orders of ϵ. To the leading order, we have

$$\lambda\left(-\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}\right)n_1+\left(\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}\right)u_1=0, \tag{20}$$

$$\lambda\left(-\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}\right)u_1-\left(\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}\right)\phi_1=0, \tag{21}$$

$$n_1=-(1+\gamma_1)\phi_1. \tag{22}$$

The solution of Eqs. (20)–(22) reads

$$\phi_1=\Phi_1(\xi,\tau)+\Phi_2(\eta,\tau), \tag{23}$$

$$n_1=-(1+\gamma_1)[\Phi_1(\xi,\tau)+\Phi_2(\eta,\tau)], \tag{24}$$

$$u_1 = -[\Phi_1(\xi,\tau) - \Phi_2(\eta,\tau)]/\lambda, \tag{25}$$

and with the solvability condition [i.e., the condition to obtain a uniquely defined n_1 and u_1 from Eqs. (20)–(22) when ϕ_1 is given by Eq. (23)], the wave velocity $\lambda^2 = 1/(1+\gamma_1)$ is also obtained. The unknown functions Φ_1 and Φ_2 will be determined at higher order. Equations (23)–(25) imply that, at the leading order, we have two waves, one of which, $\Phi_1(\xi,\tau)$, is traveling right, and the other one, $\Phi_2(\eta,\tau)$, is traveling left. At the next order, the solutions also have the following shape:

$$\phi_2 = \Psi_1(\xi,\tau) + \Psi_2(\eta,\tau), \tag{26}$$

$$n_2 = -(1+\gamma_1)[\Psi_1(\xi,\tau) + \Psi_2(\eta,\tau)], \tag{27}$$

$$u_2 = -[\Psi_1(\xi,\tau) - \Psi_2(\eta,\tau)]/\lambda, \tag{28}$$

where Ψ_1 and Ψ_2 are to be determined.

To the next higher order, we can deduce

$$\begin{aligned}\lambda \frac{\partial^2 u_3}{\partial\xi\partial\eta} &= \frac{1}{2\lambda}\frac{\partial}{\partial\xi}\left(\frac{\partial\Phi_1}{\partial\tau} + a\Phi_1\frac{\partial\Phi_1}{\partial\xi} + b\frac{\partial^3\Phi_1}{\partial\xi^3}\right)\\ &+ \frac{1}{2\lambda}\frac{\partial}{\partial\eta}\left(\frac{\partial\Phi_2}{\partial\tau} - a\Phi_2\frac{\partial\Phi_2}{\partial\eta} - b\frac{\partial^3\Phi_2}{\partial\eta^3}\right)\\ &+ \left(\frac{\partial P_0}{\partial\eta} + c\Phi_2\right)\frac{\partial^2\Phi_1}{\partial\xi^2} - \left(\frac{\partial Q_0}{\partial\xi} + c\Phi_1\right)\frac{\partial^2\Phi_2}{\partial\eta^2}\end{aligned} \tag{29}$$

or

$$\begin{aligned}\lambda u_3 &= \frac{1}{2\lambda}\int\left(\frac{\partial\Phi_1}{\partial\tau} + a\Phi_1\frac{\partial\Phi_1}{\partial\xi} + b\frac{\partial^3\Phi_1}{\partial\xi^3}\right)d\eta\\ &+ \frac{1}{2\lambda}\int\left(\frac{\partial\Phi_2}{\partial\tau} - a\Phi_2\frac{\partial\Phi_2}{\partial\eta} - b\frac{\partial^3\Phi_2}{\partial\eta^3}\right)d\xi\\ &+ \int\left(\frac{\partial P_0}{\partial\eta} + c\Phi_2\right)\frac{\partial^2\Phi_1}{\partial\xi^2}d\xi d\eta\\ &- \int\left(\frac{\partial Q_0}{\partial\xi} + c\Phi_1\right)\frac{\partial^2\Phi_2}{\partial\eta^2}d\xi d\eta,\end{aligned} \tag{30}$$

where $a = -\lambda[3 + 2\gamma_2\lambda^2 + \lambda^2 s^2(\nu\beta^2 - \mu)]/2, b = \lambda^3/2$, and $c = [1 + 4\gamma_1 - 2\gamma_2\lambda^2 - \lambda^2 s^2(\nu\beta^2 - \mu)]$.

The first (second) term in Eq. (30) will be proportional to $\eta(\xi)$ because the integrated function is independent of $\eta(\xi)$. Thus the first two terms of Eq. (30) are all secular terms, which must be eliminated in order to avoid spurious resonances. Hence we have

$$\frac{\partial\Phi_1}{\partial\tau} + a\Phi_1\frac{\partial\Phi_1}{\partial\xi} + b\frac{\partial^3\Phi_1}{\partial\xi^3} = 0, \tag{31}$$

$$\frac{\partial\Phi_2}{\partial\tau} - a\Phi_2\frac{\partial\Phi_2}{\partial\eta} - b\frac{\partial^3\Phi_2}{\partial\eta^3} = 0. \tag{32}$$

The third and fourth terms in Eq. (30) are not secular terms in this order, but they will become secular in the next order [20,22]. Hence we have

$$\frac{\partial P_0}{\partial\eta} = -c\Phi_2, \tag{33}$$

$$\frac{\partial Q_0}{\partial\xi} = -c\Phi_1. \tag{34}$$

Equations (31) and (32) are the two-side traveling wave KdV equations in the reference frames of ξ and η, respectively. Their corresponding solutions are

$$\Phi_1 = \Phi_A \operatorname{sec}^2\left[\left(\frac{a\Phi_A}{12b}\right)^{1/2}\left(\xi - \frac{1}{3}a\Phi_A\tau\right)\right], \tag{35}$$

$$\Phi_2 = \Phi_B \operatorname{sec}^2\left[\left(\frac{a\Phi_B}{12b}\right)^{1/2}\left(\eta + \frac{1}{3}a\Phi_B\tau\right)\right], \tag{36}$$

where Φ_A and Φ_B are the amplitudes of the two solitons A and B in their initial positions. The leading phase changes due to the collision can be calculated from Eqs. (33) and (34):

$$P_0 = -c\left(\frac{12b\Phi_B}{a}\right)^{1/2}\left\{\tanh\left[\left(\frac{a\Phi_B}{12b}\right)^{1/2}\left(\eta + \frac{1}{3}a\Phi_B\tau\right)\right] + 1\right\}, \tag{37}$$

$$Q_0 = -c\left(\frac{12b\Phi_A}{a}\right)^{1/2}\left\{\tanh\left[\left(\frac{a\Phi_A}{12b}\right)^{1/2}\left(\xi - \frac{1}{3}a\Phi_A\tau\right)\right] - 1\right\}. \tag{38}$$

Hence, up to $O(\epsilon^2)$, the trajectories of the two solitary waves for weak head-on interactions are

$$\begin{aligned}\xi &= \epsilon(x - \lambda t) - \epsilon^2 c\left(\frac{12b\Phi_B}{a}\right)^{1/2}\left\{\tanh\left[\left(\frac{a\Phi_B}{12b}\right)^{1/2}\right.\right.\\ &\left.\left.\times\left(\eta + \frac{1}{3}a\Phi_B\tau\right)\right] + 1\right\} + O(\epsilon^3),\end{aligned} \tag{39}$$

$$\begin{aligned}\eta &= \epsilon(x + \lambda t) - \epsilon^2 c\left(\frac{12b\Phi_A}{a}\right)^{1/2}\left\{\tanh\left[\left(\frac{a\Phi_A}{12b}\right)^{1/2}\right.\right.\\ &\left.\left.\times\left(\xi - \frac{1}{3}a\Phi_A\tau\right)\right] - 1\right\} + O(\epsilon^3).\end{aligned} \tag{40}$$

To obtain the phase shifts after a head-on collision of the two solitons, we assume that the solitons A and B are, asymptotically, far from each other at the initial time ($t = -\infty$), i.e., soliton A is at $\xi = 0, \eta = -\infty$ and soliton B is at $\eta = 0, \xi = +\infty$. After the collision ($t = +\infty$), soliton A is far to the right of soliton B, i.e., soliton A is at $\xi = 0, \eta = +\infty$ and soliton B is at $\eta = 0, \xi = -\infty$. Using Eqs. (39) and (40) we obtain the corresponding phase shifts Δ_A and Δ_B as follows:

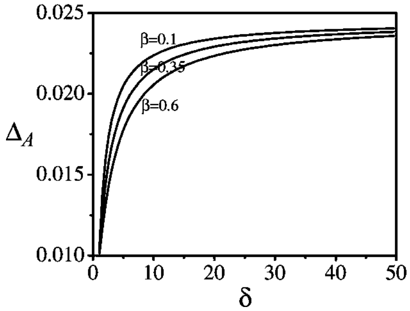

FIG. 1. Variation of phase shift with δ for different β (without dust charge variation).

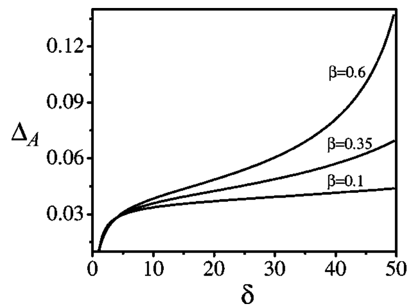

FIG. 2. Variation of phase shifts with δ for different β (with dust charge variation).

$$\Delta_A = \epsilon(x-\lambda t)|_{\xi=0,\eta=+\infty} - \epsilon(x-\lambda t)|_{\xi=0,\eta=-\infty} = 2\epsilon^2 c\left(\frac{12b\Phi_B}{a}\right)^{1/2}, \tag{41}$$

$$\Delta_B = \epsilon(x+\lambda t)|_{\eta=0,\xi=-\infty} - \epsilon(x+\lambda t)|_{\eta=0,\xi=+\infty} = -2\epsilon^2 c\left(\frac{12b\Phi_A}{a}\right)^{1/2}. \tag{42}$$

Since soliton A is traveling to the right and soliton B is traveling to the left, we see from Eqs. (41) and (42) that due to the collision each soliton has a positive phase shift in its traveling direction. The magnitudes of the phase shifts are related to the physical parameters, i.e., $\epsilon, \delta, \beta, \gamma_1, \gamma_2$, and $\Phi_A(\Phi_B)$. The variations of phase shifts with β and δ are shown in Fig. 1 (without dust charge variation, i.e., $\gamma_1 = \gamma_2 = 0$) and Fig. 2 [with dust charge variation, i.e., γ_1, γ_2 are determined by Eqs. (10) and (11)]. In all results, all physical quantities are dimensionless and $\epsilon = 0.1, \Phi_A = \Phi_B = 1$ are used. Figure 1 indicates that, for the constant dust charge case, the phase shift increases with δ but decreases with β. For given β, the phase shift increases steeply with δ when $\delta < 10$ and changes smoothly with δ when $\delta > 10$. For given δ, the phase shift decreases with β, but the change is weaker, especially when $\delta \to 1$ or $\delta \to \delta_{max}$. However, when dust charge variation is considered, the results shown in Fig. 2 are very different from those in Fig. 1. Figure 2 indicates that the phase shift increases with both δ and β. For small β (<0.1) case, the variation of phase shift with δ has a similar character as shown in Fig. 1. But the increasing rate of phase shift with δ increases with β. The increase of the phase shift with δ is steeper for larger β. For given δ, Fig. 2 shows that the phase shift increases with β when $\delta > 5$. The increase of phase shift with β is steeper for larger δ. We also can see from Fig. 1 and Fig. 2 that, for given δ and β, the phase shift with dust charge variation is larger than that without dust charge variation, and the difference is more significant for larger δ and β.

In summary, by using the extended PLK perturbation method, the leading-order analytical phase shifts of head-on collisions between two DA solitary waves in an unmagnetized dusty plasmas are derived. The effects of the plasma parameters δ and β and the dust charge variation on the phase shift are studied. It is shown that the dust charge variation has a significant effect on the phase shift. Higher-order corrections, not considered here, may give some secondary structures in the collision event, especially, for the larger wave amplitude case. This work is in progress.

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[1] P.K. Shukla and V.P. Silin, Phys. Scr. **45**, 508 (1992).
[2] N.N. Rao, P.K. Shukla, and M.Y. Yu, Planet. Space Sci. **38**, 543 (1990).
[3] F. Melandsø, Phys. Plasmas **3**, 3890 (1996).
[4] N.N. Rao, Phys. Plasmas **6**, 4414 (1999).
[5] G.E. Morfill *et al.*, Phys. Rev. Lett. **83**, 1598 (1999).
[6] D. Samsonov and J. Goree, Phys. Rev. E **59**, 1047 (1999).
[7] D.A. Law, W.H. Steel, B.M. Annaratone, and J.E. Allen, Phys. Rev. Lett. **80**, 4189 (1998).
[8] M.R. Amin, G.E. Morfill, and P.K. Shukla, Phys. Rev. E **58**, 6517 (1998).
[9] F. Verheest, Space Sci. Rev. **77**, 267 (1996).
[10] A.A. Mamun and P.K. Shukla, Phys. Scr., T **T98**, 107 (2002).
[11] P.K. Shukla, Phys. Plasmas **10**, 1619 (2003).
[12] F. Verheest, *Waves in Dusty Space Plasmas* (Kluwer, Dordrecht, 2000).
[13] P.K. Shukla and A.A. Mamun, *Introduction to Dusty Plasma Physics* (Institute of Physics Publishing, Bristol, England, 2002).
[14] A.A. Mamun and P.K. Shukla, Phys. Lett. A **290**, 173 (2001).
[15] A.A. Mamun and P.K. Shukla, Phys. Plasmas **9**, 1468 (2002).
[16] Xue Jukui and Lang He, Phys. Plasmas **10**, 339 (2003).
[17] Ju-kui Xue, Phys. Lett. A **314**, 479 (2003).
[18] Ju-kui Xue, Phys. Plasmas **10**, 3430 (2003).
[19] N.J. Zabusky and M.D. Kruskal, Phys. Rev. Lett. **15**, 240 (1965).
[20] C.H. Su and R.M. Mirie, J. Fluid Mech. **98**, 509 (1980).
[21] C.S. Gardner, J.M. Greener, M.D. Kruskal, and R.M. Miura, Phys. Rev. Lett. **19**, 1095 (1967).
[22] A. Jeffery and T. Kawahawa, *Asymptotic Methods in Nonlinear Wave Theory* (Pitman, London, 1982).

[23] Guoxing Huang and Manuel G. Velarde, Phys. Rev. E **53**, 2988 (1996).
[24] Baisong Xie, Kaifen He, and Zuoqia Huang, Phys. Lett. A **247**, 403 (1998).
[25] J.X. Ma and J. Liu, Phys. Plasmas **4**, 253 (1997).
[26] T. Taniuti and C.C. Wei, J. Phys. Soc. Jpn. **24**, 941 (1968).
[27] T. Kakutani, H. Ono, T. Taniuti, and C.C. Wei, J. Phys. Soc. Jpn. **24**, 1159 (1968).